

An invitation to algebraic topological string theory

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ABSTRACT. The purpose of this note is to provide a short invitation to the universal algebraic approach to topological string theory. In the first section we make an attempt to explain the origin of this approach and how it fits into the bigger picture of full string theory, while in the second half of this note we will introduce the relevant notions in more detail and discuss some of our main results on bulk-deformed open topological string amplitudes.

1. Introduction

We start our discussion with the bulk sector. In the low energy limit this amounts to the study of $\mathcal{N} = 1$ supergravity in ten dimensions and its solutions, viewed as vacua of closed string theory. We restrict our attention to such solutions that have a six-dimensional compact factor M . Then the Einstein equations imply that this manifold is a Calabi-Yau space.

In the framework of closed perturbative string theory these solutions M are realised as the possible targets of two-dimensional sigma models with $\mathcal{N} = (2, 2)$ superconformal symmetry. More precisely, solutions of classical gravity appear as the low energy limit of the worldsheet description. There is a distinguished set of *marginal* fields of these $\mathcal{N} = (2, 2)$ sigma models that implement infinitesimal deformations which may transform the metric of the corresponding target Calabi-Yau manifold to another Calabi-Yau space closeby. In general the superconformal field theory (CFT) \mathcal{C} associated to M is deformed to another $\mathcal{N} = (2, 2)$ CFT, which may be more “stringy” in nature in the sense that it does not need to have a geometric interpretation. The marginal fields can be viewed as defining sections of the cotangent bundle of the moduli space of $\mathcal{N} = (2, 2)$ CFTs containing our initial \mathcal{C} , namely the space of vacua continuously connected to \mathcal{C} .

More generally one can consider all *chiral primary* fields that deform away from \mathcal{C} in the much larger moduli space of $\mathcal{N} = (2, 2)$ supersymmetric field theories. Locally (anti-) chiral primaries define Riemann normal coordinates on this moduli space for a neighbourhood of the CFT we started with. We will denote these coordinates by t_i and refer to them as closed moduli.

The chiral primaries can be divided into left- and right-moving zero modes of either the (c, c) (chiral, chiral) or the (a, c) (antichiral, chiral) sectors. The first

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sector corresponds to type IIB string theory, while the second to type IIA. From the point of view of the superconformal algebra, the two types are related by an involutive outer automorphism. Thus, for every type IIA theory there should be a corresponding type IIB theory – this is precisely the mirror symmetry conjecture. Of the chiral primaries that deform the metric of the Calabi-Yau, those in the type IIB theory correspond to complex structure deformations, while those in the type IIA theory govern deformations of the complexified Kähler structure. Thus by Yau’s theorem the type IIA and type IIB theories together contain complete information about the moduli space of metrics.

Both the (c, c) and (a, c) fields, together with their operator product expansion, naturally form an algebra. In fact these are Frobenius algebras that define two topological field theories (TFTs) called the B-model and A-model, respectively. More generally one can construct two such TFTs from any $\mathcal{N} = (2, 2)$ CFT by a procedure known as topological twisting where the chiral primary fields appear as the cohomology of a BRST operator. The subsector of full string theory that builds on these TFTs is the one that we are interested in here. It is precisely the sector of all chiral primaries, and from it one can compute quantities of the *full theory* like the effective superpotential that we discuss below.

We will now consider either one of these topological twists. The associated algebra of fields ϕ_i is encoded in the structure constants $\langle \phi_i(0), \phi_j(1) \phi_k(\infty) \rangle_{\text{bulk}}$ where $\langle \cdot, \cdot \rangle_{\text{bulk}}$ is the topological metric and $0, 1, \infty$ is the standard choice of punctures on the genus zero worldsheet. Note that the set of structure constants represents only a single point in our moduli space of $\mathcal{N} = (2, 2)$ field theories. What we would like to describe is a (possibly only infinitesimal) patch around that point. The local coordinates of this patch are the closed moduli t_i , which allow us to deform the TFT *correlators* $\langle \phi_i(0), \phi_j(1) \phi_k(\infty) \rangle_{\text{bulk}}$ to the *amplitudes*

$$(1.1) \quad \left\langle \phi_i(0), \phi_j(1) \phi_k(\infty) e^{\sum_l t_l \int \phi_l^{(2)}} \right\rangle_{\text{bulk}}$$

where $\phi_l^{(2)}$ are the two-form descendants of the chiral primaries ϕ_l .

Let us expand the exponential in the amplitudes and define higher maps ℓ_n which act on the fields ϕ_i as follows:

$$\left\langle \phi_{i_0}(0), \phi_{i_1}(1) \phi_{i_2}(\infty) \int \phi_{i_3}^{(2)} \dots \int \phi_{i_n}^{(2)} \right\rangle_{\text{bulk}} = \left\langle \phi_{i_0}, \ell_n(\phi_{i_1}, \dots, \phi_{i_n}) \right\rangle_{\text{bulk}}.$$

The topological metric $\langle \cdot, \cdot \rangle_{\text{bulk}}$ together with the maps ℓ_n define what is called a Calabi-Yau L_∞ -structure [25, 26, 11] (see the next section for more details). This type of algebra completely encodes the full structure of classical closed topological string theory: knowing the maps ℓ_n and the topological metric one can compute all amplitudes.

Now we turn to the boundary sector. There are several reasons for introducing open strings, ranging from phenomenological to mathematical. For example, from a phenomenological perspective open strings and branes allow for nonabelian gauge symmetries in four-dimensional effective low energy field theories, and their presence also reduces the amount of spacetime supersymmetry. Another compelling reason for introducing open strings appears if one leaves the classical description of gravity by turning on the string coupling constant. It was in fact shown in [24] that the string coupling constant can be interpreted as the deformation parameter that quantises the moduli space of Calabi-Yau manifolds attached to a given M , hence

providing a first step to the quantisation of gravity. In this framework open strings are necessary to describe excited states of the quantum theory [20].

In the following we will restrict to the genus zero case which already has a rich structure that demands deeper understanding. We are mostly interested in boundary conditions that describe BPS branes. In the topologically twisted theory they descend to branes whose only open string states are chiral primaries that have an operator product that is strictly associative. Hence these branes and open strings naturally form the objects and morphisms of a category. Together with sewing relations and the boundary topological metric $\langle \cdot, \cdot \rangle_{\text{bdry}}$, this data combined with the closed TFT structure defines an open-closed TFT as in [14, 19]. For notational simplicity we will mostly consider the case of only one brane in this note.

The open and closed structure at one point of our moduli space of $\mathcal{N} = (2, 2)$ theories is encoded in the deformed open string three-point correlator

$$\left\langle \psi_{a_0}(p_0), \psi_{a_1}(p_1) \psi_{a_2}(p_2) \mathcal{P} e^{\sum_i u_{a_i} \int \psi_{a_i}^{(1)}} \right\rangle_{\text{bdry}}$$

which is the boundary version of the bulk sector expression (1.1). Here the ψ_a denote the open string chiral primaries inserted at generic points p_0, p_1, p_2 on the boundary of the disk, and the $\psi_a^{(1)}$ are their descendants implementing deformations of the purely open theory. The u_a are free moduli which are to be viewed as local coordinates of a non-commutative space; setting them to zero we obtain the TFT structure constants.

Just like in the bulk sector we can expand the exponential and consider individual open string amplitudes which are now given in terms of *higher products* \tilde{r}_n :

$$\left\langle \psi_{a_0}(p_0), \psi_{a_1}(p_1) \psi_{a_2}(p_2) \int \psi_{a_3}^{(1)} \dots \int \psi_{a_n}^{(1)} \right\rangle_{\text{bdry}} = \left\langle \psi_{a_0}, \tilde{r}_n(\psi_{a_1} \otimes \dots \otimes \psi_{a_n}) \right\rangle_{\text{bdry}}.$$

The topological metric $\langle \cdot, \cdot \rangle_{\text{bdry}}$ and the maps \tilde{r}_n define a *Calabi-Yau A_∞ -algebra* which arises from the symmetries of amplitudes and which completely encodes the full structure of open topological string theory [9, 7] (see the next section for the definition). Similarly, for arbitrarily many branes we are led to a Calabi-Yau A_∞ -category. In the case of the *B-model* with target M this category is the bounded derived category of coherent sheaves $\mathbb{D}^b(M)$, while the boundary sector of the *A-model* is the derived Fukaya category $\text{Fuk}(M)$.

Recall that in the purely closed sector the mirror symmetry conjecture says that there is an involution \mathcal{I} acting on the set of moduli spaces of Calabi-Yau manifolds, exchanging complex structure deformations with complexified Kähler structure deformations. In the boundary sector the conjecture extends to homological mirror symmetry which in particular states that $\mathbb{D}^b(M)$ and $\text{Fuk}(\mathcal{I}(M))$ are equivalent as Calabi-Yau A_∞ -categories. Thus we have identified a second important reason to study A_∞ -algebras.

So far we have sketched how the structure of the full moduli space of an open-closed topological string theory at genus zero consists of A_∞ -categories fibred over the commutative moduli space of closed TFTs. In the next section we will give a prescription for how to glue these fibres in a neighbourhood of a chosen point in moduli space. In particular we will arrive at an explicit recursive formula for the *bulk-deformed open string amplitudes*

$$\left\langle \psi_{a_0}(p_0), \psi_{a_1}(p_1) \psi_{a_2}(p_2) \int \psi_{a_3}^{(1)} \dots \int \psi_{a_n}^{(1)} e^{\sum_i t_i \int \phi_i^{(2)}} \right\rangle_{\text{bdry}}$$

which we will express as $\langle \psi_{a_0}, \tilde{r}_n^t(\psi_{a_1} \otimes \dots \otimes \psi_{a_n}) \rangle_{\text{bdry}}$ in terms of the higher products \tilde{r}_n^t of a curved A_∞ -algebra. Once these amplitudes are computed one immediately obtains their generating function

$$(1.2) \quad \mathcal{W}_{\text{eff}}(t, u) = \sum_{n \geq 2} \frac{1}{n+1} \left\langle \psi_{a_0}, \tilde{r}_n^t(\psi_{a_1} \otimes \dots \otimes \psi_{a_n}) \right\rangle_{\text{bdry}} u_{a_0} u_{a_1} \dots u_{a_n}$$

which is also the F-term D-brane superpotential of the effective four-dimensional low energy field theory.

Our construction splits into two parts, the first of which is restricted to the case of B-twisted affine Landau Ginzburg models (with arbitrary potential W), while the second part is completely general. Let us mention some of the reasons why Landau-Ginzburg models are interesting theories to study in this context. Firstly, the simple description of their boundary conditions in terms of matrix factorisations allows both for very explicit calculations and for a direct analysis of the underlying structure, unfettered by unnecessary complications. Secondly, by the CY/LG correspondence there is an equivalence between orbifolded Landau-Ginzburg models with quasi-homogeneous potential W and B-models whose compact targets are hypersurfaces $\{W = 0\}$ in projective space, both in the bulk [23] and boundary sector [8]. Finally, Landau-Ginzburg models are important since via RG flow they are expected to be related to full CFTs. Several quantities of interest are invariants under the flow and can thus be studied on the often more accessible Landau-Ginzburg side of this CFT/LG correspondence. Again we stress that the CFTs describable in this way can but need not have a geometric interpretation, thus covering a larger class of possible string vacua.

Finally we comment on the relation between the algebraic approach advocated here and other approaches to topological string theory. These are more geometric in nature, both in the sense that they apply only to sigma models with a certain class of target manifolds M and that they derive the effective superpotential \mathcal{W}_{eff} as a geometric quantity. In fact \mathcal{W}_{eff} is treated as a commutative function of marginal fields only, and the focus is on computing this function. There are two related geometric approaches which we briefly discuss in turn.

The most widely used approach aims to generalise the special geometry of the closed string sector [3] to the open and closed B-model, initiated in [17, 22]. While a general proof is lacking, in this approach \mathcal{W}_{eff} is computed as a linear combination of relative period integrals over 3-cycles ending on D2-branes. The computation of these period integrals involves equations of Picard-Fuchs type, and once these are solved the challenge is to find the initial conditions that will yield the correct linear combination of relative periods. This approach works for general toric targets, and it has been applied successfully to several examples with compact Calabi-Yau manifolds.

A second approach [1, 2] uses the CY/LG correspondence and is restricted to a subclass of compact Calabi-Yau manifolds. It assumes knowledge of an explicit family (parametrised by open moduli) of BPS D2-branes in the associated B-model. This family is then transported to the Landau-Ginzburg side where \mathcal{W}_{eff} can be computed to first order in the bulk moduli simply by computing TFT three-point correlators. By iterating this procedure one may obtain a complete non-perturbative description of the open string moduli space attached to the initial family of D2-branes.

The geometric approaches to computing \mathcal{W}_{eff} are very successful and efficient for the classes of models they are applicable to. This level of efficiency has not yet been achieved in the approach via homotopy algebras, which is however mostly due to the fact that it has received much less attention so far. On the other hand, the algebraic approach is founded on the symmetries of amplitudes and thus universally applies to any topological string theory. As we will see below the A_∞ -structure encoding the amplitudes is derived directly from a Chern-Simons-esque string field theory. In this sense it is also more conceptual, and effective superpotentials (1.2) are obtained as a byproduct.

2. Algebraic approach to bulk-deformed open topological string theory

In this section we start with a very short introduction to A_∞ - and L_∞ -algebras. Then we discuss B-twisted Landau-Ginzburg models and explain how such algebras describe their boundary and bulk sectors, respectively. Finally, we review the construction of bulk-induced deformations of open topological string theory for such models. In an attempt to hide less significant technical details from this exposition, we shall treat signs, degrees, shifts etc. rather negligently. For a full account we refer to our paper [6].

2.1. A_∞ - and L_∞ -basics. Let us start by recalling the necessary background on *algebras with higher structures*. Most importantly, a *curved A_∞ -algebra* is a graded vector space A together with an operator ∂ on the space $T_A = \bigoplus_{n \geq 0} A^{\otimes n}$ satisfying (i) $\Delta\partial = (\partial \otimes 1 + 1 \otimes \partial)\Delta$ and (ii) $\partial^2 = 0$. Here Δ is the comultiplication defined by $\Delta(a_1 \otimes \dots \otimes a_n) = \sum_{j=0}^n (a_1 \otimes \dots \otimes a_j) \otimes (a_{j+1} \otimes \dots \otimes a_n)$, and property (i) tells us that ∂ is a *coderivation*, i.e. the dual notion of a derivation. One can show that it is completely determined by the family of *higher products*

$$r_n = \pi_A \partial|_{A^{\otimes n}} : A^{\otimes n} \longrightarrow A, \quad n \geq 0,$$

where π_A is the projection $T_A \rightarrow A$. Accordingly we will use both (A, ∂) and (A, r_n) to denote A_∞ -algebras. The maps r_n are constrained by quadratic relations coming from property (ii). In terms of $C = r_0(1)$, $d = r_1$ and $a \cdot b = \pm r_2(a \otimes b)$ the constraint $\partial^2 = 0$ in particular implies $d(C) = 0$, $d^2(a) = a \cdot C - C \cdot a$ and the product rule $d(a \cdot b) = d(a) \cdot b \pm a \cdot d(b)$. In the special case when $r_n = 0$ for all $n \geq 3$ the only additional constraint is that the product r_2 is associative, and hence the data (A, C, d, \cdot) define a *curved differential graded (DG) algebra*.

From the above we see that for any curved A_∞ -algebra r_1 is a differential (to be interpreted as the BRST operator in topological string theory) whenever the *curvature* r_0 is central or vanishes. We call the A_∞ -algebra *minimal* if the differential vanishes too. Furthermore, (A, r_n) is *cyclic* with respect to a pairing $\langle \cdot, \cdot \rangle$ on A if $\langle a_0, r_n(a_1 \otimes \dots \otimes a_n) \rangle = \pm \langle a_n, r_n(a_2 \otimes \dots \otimes a_n \otimes a_0) \rangle$ for all $n \geq 0$. Finally, an A_∞ -algebra is *Calabi-Yau* if it is minimal, cyclic with respect to a non-degenerate pairing, and has a unit compatible with the higher products. As recalled in the previous section, every open topological string theory is described by a Calabi-Yau A_∞ -algebra.

We also need the notion of a *morphism* between curved A_∞ -algebras (A, ∂) and (A', ∂') . This is a map F from T_A to $T_{A'}$ such that $\Delta'F = (F \otimes F)\Delta$ and $F\partial = \partial'F$. Again, F is determined by maps $F_n = \pi_{A'} F|_{A^{\otimes n}}$ subject to certain compatibility conditions with the higher products coming from $F\partial = \partial'F$. We call F an *A_∞ -isomorphism* if F_1 is an isomorphism $A \rightarrow A'$. In the case of

(non-curved) A_∞ -algebras F is called an A_∞ -quasi-isomorphism if F_1 induces an isomorphism between the cohomologies of the differentials r_1 and r'_1 .

The most fundamental result on (non-curved) A_∞ -algebras is the *minimal model theorem* [10, 18]. It states that any A_∞ -algebra (A, r_n) is A_∞ -quasi-isomorphic to a minimal A_∞ -algebra $(H = H_{r_1}(A), \tilde{r}_n)$ which is unique up to A_∞ -isomorphism.

To see this explicitly, let us restrict to the case where (A, r_n) is a DG algebra and decompose $A \cong H \oplus B \oplus L$ where $B = \text{Im}(r_1)$ and L is the complement of $\text{Ker}(r_1)$. Such a decomposition provides us with a map $G = (r_1|_L)^{-1}\pi_B : A \rightarrow A$ which together with $\lambda_2 = r_2$ allows us to recursively define for $n \geq 3$:

$$\lambda_n = -r_2(G \otimes 1)(\lambda_{n-1} \otimes 1) - r_2(1 \otimes G)(1 \otimes \lambda_{n-1}) - \sum_{\substack{i, j \geq 2, \\ i+j=n}} r_2(G \otimes G)(\lambda_i \otimes \lambda_j).$$

Then the higher products on H are given by $\tilde{r}_n = \pi_H \lambda_n$, and the components of the A_∞ -quasi-isomorphism $F : (H, \tilde{r}_n) \rightarrow (A, r_n)$ are the inclusion $F_1 : H \hookrightarrow A$ and $F_n = G \lambda_n F_1$ for $n \geq 2$. Note that these formulas precisely arise from the Feynman diagrams computed in the topological string field theory (A, r_n) where G plays the role of the propagator [15].

Just like A_∞ -algebras are generalisations of associative algebras A with higher products $r_n : A^{\otimes n} \rightarrow A$, L_∞ -algebras are generalisations of Lie algebras V with higher brackets $\ell_n : V^{\wedge n} \rightarrow V$. Similarly, there are appropriate notions of L_∞ -morphisms which again can be presented as maps $L_n : V^{\wedge n} \rightarrow V'$, subject to certain compatibility conditions. The detailed definitions will not matter to us as the only L_∞ -algebras relevant in this note are *DG Lie algebras*, i.e. vector spaces endowed with a graded anti-symmetric bracket that satisfies the super Jacobi identity, and a differential compatible with the bracket. It is however crucial to view them as special L_∞ -algebras since L_∞ -morphisms (with higher components) between them will be important in what follows. The root of this fact lies in the relation between deformations and solutions to Maurer-Cartan equations that we discuss next.

Our aim is to study bulk-deformed open topological string theory. Hence we must explain what a *deformation* of an A_∞ -algebra (A, ∂) is. By definition it is a map $\delta \in \text{End}(T_A)$ such that $(A, \partial + \delta)$ is a curved A_∞ -algebra. This means that δ is a coderivation, $\delta \in \text{Coder}(T_A)$, and $(\partial + \delta)^2$ must vanish. Thus if we write $[\cdot, \cdot]$ for the graded commutator and use $\partial^2 = 0$, we see that δ must solve

$$(2.1) \quad [\partial, \delta] + \frac{1}{2}[\delta, \delta] = 0.$$

Let us recall that for any DG Lie algebra $(V, d, [\cdot, \cdot])$ its associated *Maurer-Cartan equation* reads $d(\delta) + \frac{1}{2}[\delta, \delta] = 0$. We denote its space of formal power series solutions modulo gauge transformations $\delta \mapsto \delta + d(\delta) + [\varphi, \delta]$ as $\mathcal{MC}(V, d, [\cdot, \cdot])$. It is an important result [13] that given an L_∞ -morphism $L : (V, d, [\cdot, \cdot]) \rightarrow (V', d', [\cdot, \cdot]')$, the map

$$(2.2) \quad \delta \mapsto \sum_{n \geq 1} \frac{1}{n!} L_n(\delta^{\wedge n})$$

induces a map $\mathcal{MC}(V, d, [\cdot, \cdot]) \rightarrow \mathcal{MC}(V', d', [\cdot, \cdot]')$. Furthermore, this is an isomorphism if L is an L_∞ -quasi-isomorphism.

We now make the obvious yet crucial observation that the deformation condition (2.1) is precisely the Maurer-Cartan equation of the DG Lie algebra $\text{Coder}(T_A)$

with differential $[\partial, \cdot]$ and bracket $[\cdot, \cdot]$. In addition, solving (2.1) to first order (up to gauge transformations) is identical to computing *Hochschild cohomology* $\mathrm{HH}^\bullet(A, \partial) = H_{[\partial, \cdot]}(\mathrm{Coder}(T_A))$. More importantly, the above result on transporting Maurer-Cartan solutions will allow us to construct all A_∞ -deformations if we can find a quasi-isomorphic DG Lie algebra whose Maurer-Cartan solutions are known. In the remainder of this note we will review how to do this for the case of Landau-Ginzburg models, where the known Maurer-Cartan solutions will be precisely the space of bulk fields.

2.2. Landau-Ginzburg models. We shall consider the topological B-twist of Landau-Ginzburg models with affine target $X = \mathbb{C}^N$ and potential $W \in R = \mathbb{C}[x_1, \dots, x_N]$. The *on-shell* space of states in the bulk sector of such two-dimensional topological field theories is the Jacobian $\mathrm{Jac}(W) = R/(\partial_1 W, \dots, \partial_N W)$ [21]. This is obtained from the *off-shell* bulk space of polyvector fields $T_{\mathrm{poly}} = \Gamma(X, \bigwedge T^{(1,0)}X)$ by taking cohomology with respect to the BRST operator $[-W, \cdot]_{\mathrm{SN}}$. Here we denote by $[\cdot, \cdot]_{\mathrm{SN}}$ the Schouten-Nijenhuis bracket, the extension of the Lie bracket to polyvector fields. Note that T_{poly} together with the BRST differential and the bracket has the structure of a DG Lie algebra. A direct computation shows that its Maurer-Cartan solutions are precisely the on-shell bulk space $\mathrm{Jac}(W)$.

The boundary sector is defined by matrix factorisations of W , i.e. odd supermatrices D with polynomial entries such that D squares to $W \cdot \mathrm{id}$ [12, 4, 16]. For simplicity we will only consider one such boundary condition. Then the *off-shell* space A is simply the space of all polynomial matrices of the same size as D . The boundary BRST operator is given by the graded commutator $[D, \cdot]$, and together with matrix multiplication this makes A a DG algebra (A, ∂) . By definition its cohomology is the *on-shell* space H .

Since the off-shell algebra (A, ∂) is an A_∞ -algebra (describing open topological string field theory for Landau-Ginzburg models), we can apply the minimal model theorem to produce an A_∞ -structure $\tilde{\partial}$ on H together with an A_∞ -quasi-isomorphism $F : (H, \tilde{\partial}) \rightarrow (A, \partial)$. For generic choices of propagators $(H, \tilde{\partial})$ will not be Calabi-Yau, but this can be corrected using methods of non-commutative geometry as explained in [5]. Hence we can assume that for any Landau-Ginzburg model and all their branes we can construct a Calabi-Yau A_∞ -algebra $(H, \tilde{\partial})$ that encodes the full structure of open topological string theory.

2.3. Bulk-deformed Landau-Ginzburg models. Now we discuss bulk deformations of $(H, \tilde{\partial})$. We saw that any deformation must solve the Maurer-Cartan equation of $(\mathrm{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot])$, but finding such solutions directly is hard. However, we are only interested in *bulk-induced deformations* which we define as the image of on-shell bulk fields under an L_∞ -morphism

$$(2.3) \quad (T_{\mathrm{poly}}, [-W, \cdot]_{\mathrm{SN}}, [\cdot, \cdot]_{\mathrm{SN}}) \longrightarrow (\mathrm{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot]).$$

The left-hand side is the off-shell bulk algebra, and we already know that its Maurer-Cartan solutions precisely form the on-shell bulk space $\mathrm{Jac}(W)$. Hence our remaining task is to construct the L_∞ -map (2.3). We shall do this in two main steps which are Theorems 1 and 2 below.

To set the stage, let us write the A_∞ -structure of the off-shell open string algebra (A, ∂) as $\partial = \partial_1 + \partial_2$ to emphasise that it is a DG algebra with differential

$r_1 = [D, \cdot]$ and matrix multiplication r_2 . There is another natural curved A_∞ -structure on A which we write as $(A, \partial_0 + \partial_2)$, where the coderivation ∂_0 corresponds to the curvature $r_0 = -W \cdot \text{id}$. Similarly, the polynomial ring R is also a curved algebra $(R, \hat{\partial}_0 + \hat{\partial}_2)$, where $\hat{\partial}_0$ corresponds to multiplication with $-W$ and $\hat{\partial}_2$ to the usual product. Now we can take the first step:

Theorem 1 ([6]). *There is a sequence of explicit L_∞ -quasi-isomorphisms*

$$\begin{aligned} (T_{\text{poly}}, [-W, \cdot]_{\text{SN}}, [\cdot, \cdot]_{\text{SN}}) &\xrightarrow{\varphi_1} (\text{Coder}(T_R), [\hat{\partial}_0 + \hat{\partial}_2, \cdot], [\cdot, \cdot]) \\ &\xrightarrow{\varphi_2} (\text{Coder}(T_A), [\partial_0 + \partial_2, \cdot], [\cdot, \cdot]) \\ &\xrightarrow{\varphi_3} (\text{Coder}(T_A), [\partial_1 + \partial_2, \cdot], [\cdot, \cdot]). \end{aligned}$$

The map φ_3 is simply the adjoint action of the A_∞ -isomorphism T (which “cancels” the “tadpole” r_0) whose non-vanishing components are $T_0 = D$ and $T_1 = 1_A$, and φ_2 is Morita equivalence for curved DG-algebras.

It turns out that the map φ_1 deserves more attention; it is an interesting variant of deformation quantisation. We recall that the latter is a method to quantise the algebra of classical observables $C^\infty(M, \mathbb{R})$ on a phase space M , which we take to be \mathbb{R}^d . The idea is to deform the commutative, associative multiplication \hat{r}_2 on $C^\infty(M, \mathbb{R})$ to an associative but non-commutative \star -product of quantum observables. This deformation again leads to a Maurer-Cartan equation. Kontsevich’s solution [13] was to explicitly construct an L_∞ -quasi-isomorphism $K : (\Gamma(M, \wedge TM), [\cdot, \cdot]_{\text{SN}}) \rightarrow (\text{Coder}(T_{C^\infty(M, \mathbb{R})}), [\hat{\partial}_2, \cdot], [\cdot, \cdot])$, thus providing a one-to-one correspondence between Poisson structures on M and perturbative \star -products of quantum observables.

Note that the map K is precisely the special case of our map φ_1 if $W = 0$. Indeed, we can show that K continues to be an L_∞ -map after “turning on” W . Furthermore, it also remains a quasi-isomorphism as a direct computation yields $\text{HH}^\bullet(R, \hat{\partial}_0 + \hat{\partial}_2) \cong \text{Jac}(W)$. Hence we arrive at a generalisation of deformation quantisation, namely that the DG Lie algebra of polyvector fields with differential $[-W, \cdot]$ governs deformations of $(R, \hat{\partial}_0 + \hat{\partial}_2)$.

With Theorem 1 we have explicitly classified all deformations of the off-shell open string algebra (A, ∂) , and we found that all such deformations are bulk-induced. We stress that this off-shell, string field theoretic result is important by itself as it contains information on descendants of ground states. However, one is also interested in computing amplitudes and effective superpotentials in on-shell open topological string theory. Accordingly, we will discuss how to transport deformations of (A, ∂) to those of $(H, \tilde{\partial})$. Again, this will be achieved by an L_∞ -morphism that maps off-shell deformations on-shell via (2.2):

Theorem 2 ([6]). *There is an explicit L_∞ -map*

$$(\text{Coder}(T_A), [\partial, \cdot], [\cdot, \cdot]) \longrightarrow (\text{Coder}(T_H), [\tilde{\partial}, \cdot], [\cdot, \cdot]).$$

The main tool that allows us to construct this map is an L_∞ -version of the *homological perturbation lemma*, which is another structure transfer result. Given two complexes (C_1, d_1) , (C_2, d_2) with morphisms $i : (C_2, d_2) \rightarrow (C_1, d_1)$, $p : (C_1, d_1) \rightarrow (C_2, d_2)$ and $h \in \text{End}(C_1)$, we call these data a deformation retraction if $pi = 1_{C_2}$

and $1_{C_1} - ip = d_1 h + h d_1$. In this situation the perturbation lemma states that

$$(2.4) \quad \delta \mapsto \sum_{n \geq 1} p(\delta h)^n \delta i$$

maps a deformation δ of (C_1, d_1) to a deformation of (C_2, d_2) , and this can be understood in terms of an L_∞ -quasi-isomorphism between the endomorphism algebras of the two complexes.

We now use this method in our setting. It was shown in [6] that for any A_∞ -algebra (A, ∂) with minimal model $(H, \tilde{\partial})$ there is a deformation retraction

$$(T_H, \tilde{\partial}) \xrightleftharpoons[\bar{F}]{F} (T_A, \partial) \xleftarrow{U} \cdot$$

Here F is our usual minimal model quasi-isomorphism, and \bar{F}, U are defined recursively by

$$U_n = -\frac{1}{2} Gr_2 \left(\sum_{i=1}^{n-1} (U_i \otimes (1_{T_A} + F\bar{F})_{n-i} + (1_{T_A} + F\bar{F})_{n-i} \otimes U_i) \right),$$

$$\bar{F}_n = -\frac{1}{2} \pi_H r_2 \left(\sum_{i=1}^{n-1} (U_i \otimes (1_{T_A} + F\bar{F})_{n-i} + (1_{T_A} + F\bar{F})_{n-i} \otimes U_i) \right).$$

Furthermore, these maps are such that (2.4) maps coderivations to coderivations.

Now we have everything in place to apply the above to the case where (A, ∂) and $(H, \tilde{\partial})$ are the off-shell and on-shell open string algebras. It follows from Theorem 1 that all deformations δ of (A, ∂) are coderivations determined by $1 \mapsto \sum_i t_i \phi_i$ where ϕ_i are on-shell bulk fields in $\text{Jac}(W)$ and t_i are the closed moduli. Then Theorem 2 tells us that the associated deformations $\tilde{\delta}$ of the on-shell algebra $(H, \tilde{\partial})$ are given by

$$\tilde{\delta} = \sum_{n \geq 1} \bar{F}(\delta U)^n \delta F.$$

All the ingredients F, \bar{F}, U can be algorithmically computed, thus leading to completely explicit expressions for the bulk-deformed open string amplitudes

$$\begin{aligned} & \left\langle \psi_{a_0}, \tilde{r}_n^t(\psi_{a_1} \otimes \dots \otimes \psi_{a_n}) \right\rangle_{\text{bdry}} \\ &= \left\langle \psi_{a_0}(p_0), \psi_{a_1}(p_1) \psi_{a_2}(p_2) \int \psi_{a_3}^{(1)} \dots \int \psi_{a_n}^{(1)} e^{\sum_i t_i \int \phi_i^{(2)}} \right\rangle_{\text{bdry}} \end{aligned}$$

where the higher products \tilde{r}_n^t are the components of $\tilde{\partial} + \tilde{\delta}$.

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